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NOTE ON THE COWLING MODEL OF A CONVECTIVE-RADIATIVE STAR*

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ABSTRACT. The question how far the simple Cowling model of a star of the convective-radiative type conforms to Bethe's law of energy generation, is examined from different points of view, and it is also indicated how an agreement of the Cowling model with Bethe's law may be best achieved.

Since the appearance of Bethe's law of energy generation (1939) in stellar interiors, it has become clear that energy is generated in the main sequence stars within a small core around the centre. Attention has been directed to the construction of stellar models with an energy generating core in convective equilibrium, surrounded by a radiative envelope. Cowling (1935) constructed a stellar model of this nature ; it is a point source model with a central convective core, surrounded by an envelope in which the transfer of energy is entirely by radiation. The effect of radiation pressure being assumed to be negligible, the core is an Emden polytrope of index $n=3/2$. The principal characteristics of the model are the following :—

(*i*) the core occupies 16·9 p.c. of the total radius and encloses 14·5 p.c. of the total mass ;

(*ii*) the radiative temperature gradient in the envelope is given by

$$\frac{d\theta}{d\xi} = -Q\frac{\sigma^2\theta^{-6\cdot5}}{\xi^2}$$

with
$$Q=\frac{\kappa_0 L}{16\pi}\frac{3}{ac}\frac{\rho_c^2}{\alpha T_c^{7\cdot5}}=0\cdot1968 \text{ (a numerical constant) ;} \quad (1)$$

$$\alpha=\left(\frac{5k}{8\pi\mu GH}\frac{T_c}{\rho_c}\right)^{1/2},$$

L = total luminosity, κ_0 = constant in Kramers' opacity formula $\kappa=\kappa_0\,\rho/T^{3\cdot5}$, which is assumed to hold here ; ξ, σ and θ are respectively central distance, density and temperature in suitable units.

(*iii*) the mass, radius, central temperature and the central density are connected by the following formulae—

$$T_c = 0\cdot9\,\frac{\mu H}{k}\frac{GM}{R}, \qquad (2)$$

$$\rho_c = \frac{\xi_1^3}{4\pi\psi_1}\frac{M}{R^3},$$

where $\xi_1=7\cdot027$, denotes the boundary of the configuration, and

$$\psi_1=\left(-\xi^2\frac{d\theta}{d\xi}\right)_{\xi=\xi_1}=3\cdot1237.$$

This simple model is of considerable importance in view of Bethe's law of energy generation, which demands a stellar model to be of the convective-

* A short note on the subject has been published in *Nature*, 153, 166, 1944.

radiative type. It would be useful to examine how far this model is able to predict any of the 3 observed quantities L, M and R of a star, and to compare the luminosity furnished by this model with that calculated from Bethe's formula. We give here some calculations from three different points of view.

We first note that there are 7 parameters occuring in the above equations, viz., L, M, R, κ_0, T_c, ρ_c and μ which are of course not all independent. κ_0 is connected with the H_2-content X, and hence with the mean molecular weight μ by the relations

$$\kappa_0 = 3 \cdot 9 \times 10^{25} \, (1 - X^2) \frac{1}{t}, \quad \mu = \frac{2}{1+3X} \tag{3}$$

(He-content being assumed negligible), where t denotes the average guillotine factor. The value of t will be an uncertainty in our calculations. A limiting value t = 2, as taken by Chandrasekhar in his theory of stellar envelopes, will be rather low for our purpose since the averaging should extend over the entire radiative envelope. Near the interface where the temperature is quite high, the guillotine factor has certainly a much higher value. In our calculations we have therefore used two values of t, (*i*) t = 2 (an extreme case) and (*ii*) t = some higher value appropriate to the temperature and density conditions.

Case 1. We take the observed values of L, M, R of a star, and obtain from equations (1), (2) T_c, ρ_c, μ and hence by (3) X. For instance, for the

(i) *Sun* ($L = L_\odot$, $M = \odot$, $R = R_\odot$), we obtain, taking t = 2,

$T_c = 24 \cdot 4 \times 10^6$ degrees, $\rho_c = 52 \cdot 3$ gm/cm^3, $\mu = 1 \cdot 175$, $X \backsim 0 \cdot 22$.

For the higher value of t, we take t = 5 as suggested by Strömgren. The corresponding values are:

$T_c = 21 \cdot 4 \times 10^6$ degrees, $\rho_c = 52 \cdot 3$ gm/cm^3, $\mu = 1 \cdot 028$, $X \backsim 0 \cdot 31$.

(*ii*) η – *Cassiopeiae A* ($L = 0 \cdot 83 \, L\odot$, $M = 0 \cdot 72 \odot$, $R = 0 \cdot 83 \, R_\odot$) t = 2 gives $T_c = 26 \cdot 1 \times 10^6$ degrees, $\rho_c = 65 \cdot 8$ gm/cm^3, $\mu = 1 \cdot 443$, $X \backsim 0 \cdot 13$, t = 5 gives $T_c = 23 \cdot 1 \times 10^6$ degrees, $\rho_c = 65 \cdot 8$ gm/cm^3, $\mu = 1 \cdot 273$, $X \backsim 0 \cdot 19$.

Evidently t = 5 gives more acceptable results in these two cases. From Chandrasekhar's theory of stellar envelopes it is known that the influence of radiation pressure in η-Casseopeiae A is small, and so comparison with the Cowling model is justified. These calculated central values in themselves not improbable, will however lead to a serious discrepancy. If the stellar equations are integrated from the centre outwards with these central conditions, and the luminosity (required to determine the change to radiative gradient) calculated by Bethe's formula, we arrive at an all-convective model in each case, there being no radiative envelope. This can be verified by calculations; a theoretical proof (for the general case) and other connected results will be given in a different communication.

Case 2. Here we prescribe M, R and μ. Then equations (2) give T_c, ρ_c and hence (1) gives L which we call L_C. On the other hand the luminosity may be calculated by Bethe's formula, using the above values of T_c, ρ_c and the results of integration; this value we denote by L_B. A comparison of these two values of L will give us an idea regarding the compatibility of the Cowling model with Bethe's law of energy generation. For instance, for the *Sun* (with $\mu=1$, $X=0{\cdot}35$, $t=2$) we obtain on the above basis

$T_c = 20{\cdot}8 \times 10^6$ degrees, $\rho_c = 52{\cdot}3$ gm/cm^3.

$L_C = 1{\cdot}2 \times 10^{33}$ ergs/sec., $L_B = 6{\cdot}9 \times 10^{33}$ ergs/sec.

For $t=5$ the only modified value is $L_C = 3{\cdot}0 \times 10^{33}$ ergs/sec.
The discrepancy in the value of L arising out of the use of Bethe's formula in the Cowling model is by a factor between 2 and 6 (for t varying from 5 to 2) which may be regarded as fairly satisfactory. But here also the discrepancy is high enough to yield an all-convective model, if integrated from inside as stated in case 1.

Case 3. Here we assign μ and then choose T_c, ρ_c in such a manner that L_C calculated from (1) is identical with L_B calculated from Bethe's formula and both equal to say, the solar luminosity. Then equations (2) will give the values of M and R which may be compared with the observed solar values. We briefly give the method of this calculation.

The equation for the luminosity-determination is

$$L(r) = \int_0^r 4\pi\rho r^2 \varepsilon\, dr$$

where $\varepsilon = \varepsilon_0\, \rho\, T^{-2/3} e^{-B/T^{1/3}}$ (Bethe's formula),

ε_0 is a constant depending on the composition, and $B = 56 \times (2 \times 10^7)^{1/3}$.
Introducing the variables ξ, σ and θ defined by $r = \alpha\xi$, $\rho = \rho_c\sigma$ and $T = T_c\theta$,

where $$\alpha = \left(5k/8\pi\mu GH \frac{T_c}{\rho_c}\right)^{1/2}$$

and using the polytropic relation $\sigma = \theta^{3/2} (n = 3/2)$, we obtain after some calculations

$$L(\xi) = A\, \rho_c^{1/2}\, I(\xi, T_c) \tag{4}$$

where $$A = 4\pi\varepsilon_0 \left(\frac{5k}{8\pi\mu GH}\right)^{3/2} T_c^{5/6}$$

and $$I(\xi, T_c) = \int_0^\xi \theta^{7/3} e^{-b/\theta^{1/3}} \xi^2\, d\xi \quad (b = B/T_c^{1/3}).$$

For a given T_c, the constant A and the integral $I(\xi, T_c)$ can be evaluated and hence $L(\xi)$ determined as a function of ρ_c only. Also the luminosity can be calculated from (1) and expressed in the form

$$L = Q\frac{16\pi ac}{3\kappa_0}\left(\frac{5k}{8\pi\mu GH}\right)^{1/2} \frac{T_c^{8}}{\rho_c^{5/2}} = B/\rho_c^{5/2} \tag{5}$$

where $$B = Q\frac{16\pi ac}{3\kappa_0}\left(\frac{5k}{8\pi\mu GH}\right)^{1/2} T_c^{8}.$$

Hence for the agreement between the two values of L given by (4) and (5) we must have at the interface $\xi = \xi_i$ of the convective core

$$A\, \rho_c^{1/2}\, I(\xi_i, T_c) = B/\rho_c^{5/2}$$

or $$\rho_c^3 = B/A\,I(\xi_i, T_c). \tag{6}$$

This equation determines ρ_c when T_c is assigned. By a little trial T_c, ρ_c

may be so adjusted that the calculated luminosity becomes identical with the luminosity of the given star.

Taking $\mu = 1$, $X = 0{\cdot}35$, for the Sun, and $t = 2$, we find by a little trial that $T_c = 20{\cdot}5 \times 10^6$ degrees, $\rho_c = 31{\cdot}6$ gm/cm^3, satisfy the above conditions, and yield the solar luminosity $L = 3{\cdot}8 \times 10^{33}$ ergs/sec. The corresponding mass and radius, calculated from (2) come out to be $M = 2{\cdot}5 \times 10^{33}$ gm, $R = 8{\cdot}8 \times 10^{10}$ cm. For $t = 5$ the results are further improved, and we obtain $M = 2{\cdot}1 \times 10^{33}$ gm, $R = 7{\cdot}6 \times 10^{10}$ cm, the corresponding values of T_c and ρ_c being $20{\cdot}3 \times 10^6$ degrees and 43 gm/cm^3 respectively. The very close agreement in respect of mass and radius for $t = 5$ is indeed remarkable. It may be noted that the convective-radiative character of the model is not disturbed by these central conditions (as happened in Cases 1 and 2). If we perform the same calculations in the case of the η-Cassiopeiae A, taking $\mu = 1$, $X = 0{\cdot}35$ and $t = 5$, we obtain

$$T_c = 20{\cdot}0 \times 10^6 \text{ degrees}, \quad \rho_c = 46{\cdot}4 \text{ gm/cm}^3, \quad M = 1{\cdot}9 \times 10^{33} \text{ gm}$$

$$R = 7{\cdot}2 \times 10^{10} \text{ cm}, \quad L_C = L_B = 3{\cdot}0 \times 10^{33} \text{ ergs/sec.}$$

In this case the agreement is not as good as in the case of the Sun. We have of course assumed the same H_2-concentration as for the Sun ; a slightly higher H_2-concentration for η-Casseopeiae A will very much improve the results, while a lower concentration will make the discrepancy worse. It appears possible to so choose the hydrogen concentration that a close agreement between the observed and calculated values (by the Cowling model) may be achieved.

From these calculations it appears that the agreement of the Cowling model with Bethe's formula for the Sun will be the best and very satisfactory, if the central conditions are so chosen that with a plausible value of the average guillotine factor, the calculated and observed luminosities become identical. It appears possible that for the main sequence stars of small masses, by an adjustment of the H_2-content near about $X \sim 0{\cdot}35$, a close agreement between calculated and observed stellar parameters may be obtained in many cases. One may thus form an estimate of the H_2-content in these stars on the basis of the Cowling model and Bethe's energy generation formula. It is also permissible to suggest, that a main sequence stellar model with approximately the solar mass, and of the convective-radiative type in agreement with Bethe's energy generation formula, should have a central temperature of the order of twenty million degrees as suggested by the standard model, but the central density should be very much lower than that of the standard model, if the idea of an H_2-content of about $X \sim 0{\cdot}35$ is to be retained.

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For an illuminating account of the Cowling model, see also Chandrasekhar's "Introduction to the study of Stellar Structure", p. 351.